MATH-329 Nonlinear optimization
Exercise session 10: KKT

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Document compiled on November 20, 2024

1. KKT. Consider the set

$$S = \{(x, y) \mid 0 \leq x \leq 2\pi \text{ and } 0 \leq y \leq \sin(x) + 2\}$$

and the cost function $f(x, y) = -y$ that we want to minimize.

1. Draw the search space.
2. Is the problem convex?
3. Do CQs hold globally?
4. Write down the KKT conditions.
5. Find all KKT points and all stationary points.
6. Find all local and global minima.
7. How does your answer change if the set is

$$S = \{(x, y) \mid 0 \leq x \leq 2\pi \text{ and } 0 \leq y \leq \sin(x) + 1\}$$

instead? In particular, is LICQ or MFCQ holding at the point $(3\pi/2, 0)$?

To train you may want to revisit this exercise for a general linear cost function $f(x, y) = x\cos(\theta) + y\sin(\theta)$ with parameter $\theta \in [0, 2\pi[$.

2. KKT (bis). Consider the problem

$$\min_{x,y \in \mathbb{R}} -2x + y \qquad \text{subject to} \qquad \begin{cases} (1-x)^3 - y & \geq 0 \\ y + \frac{1}{4}x^2 - 1 & \geq 0. \end{cases}$$

The optimal solution is $x^\star = (0, 1)^\top$, where both constraints are active.

1. Do CQs hold at $x^\star$?
2. Is $x^\star$ stationary?
3. Is $x^\star$ a KKT point?

Answer the same questions for the optimization problem

$$\min_{x,y \in \mathbb{R}} y \quad \text{subject to} \quad x^2 - y^3 \leq 0,$$

where the global minimum is $x^\star = (0, 0)^\top$.

3. Convex constraints. In the lecture we showed that the set

$$S = \{x \in \mathcal{E} \mid h_i(x) = 0, i = 1, \ldots, p, \text{ and } g_i(x) \leq 0, i = 1, \ldots, m\}$$

is convex if the functions $h_1, \ldots, h_p$ are affine and the functions $g_1, \ldots, g_m$ are convex.

1. Provide an example of a non-convex set S where all the functions $g_1, \ldots, g_m$ are convex but one function h_i is not affine.

2. Provide an example of a non-convex set S where all the functions $h_1, \ldots, h_p$ are affine but one function g_i is non-convex.

4. Quadratic constraints. Sometimes quadratically constrained programs can be solved efficiently.

1. We let $S = \{x \in \mathbb{R}^n \mid x^\top Ax + b^\top x + c \leq 0\}$ where $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n, c \in \mathbb{R}$. Show that S is convex if $A \succeq 0$.

2. Suppose $A \succ 0$ and let $d \in \mathbb{R}^n$. Find a solution for the minimization problem

$$\min_{x\in S} d^\top x$$

where $S = \{x \in \mathbb{R}^n \mid x^\top Ax \leq 1\}$ and $A \succ 0$.